ON A BOUNDARY VALUE PROBLEM OF THE THEORY OF OSCILLATIONS WITH PARAMETER-DEPENDENT BOUNDARY CONDITIONS

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A homogeneous second order differential equation with homogeneous boundary conditions dependent on the parameter, is investigated. Such an equation is obtained in the course of solution of the problem of characteristic oscillations of an ideal incompressible fluid in an elastic vessel, when the method of separation of variables is used. We prove the completeness of the system of eigenfunctions of our boundary value problem and we derive the expansion of an arbitrary, piecewise-continuous function into a series in terms of these eigenfunctions.

1. Given the boundary value problem

$$\frac{d^2y}{dx^2}+\lambda^2y=0 \qquad x=0, \quad (A_0+A_1\lambda^2+A_2\lambda^4)\frac{dy}{dx}=y; \qquad x=1, \quad B\frac{dy}{dx}=y \qquad (1.1)$$

we have to find the eigenvalues λ_n, the eigenfunctions $y_n(x, \lambda)$, show that the set of these eigenfunctions is complete, and construct the expansion of an arbitrary, piecewise-continuous bounded function $f(x)$ into a Fourier series in terms of $y_n(x, \lambda)$.

2. To solve the above problem we shall, following [1], first consider vibrations of a string

$$\frac{\partial^2 y}{\partial \xi^2}-\frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}=0 \qquad (2.1)$$

with specially chosen boundary conditions

$$\xi=0, \quad \left(A_0'-A_1'\frac{\partial^2}{\partial t^2}+A_2'\frac{\partial^4}{\partial t^4}\right)\frac{\partial y}{\partial \xi}=y; \qquad \xi=l, \quad B'\frac{\partial y}{\partial \xi}=y \qquad (2.2)$$

Here y is the deflection of the string, ξ is the coordinate of the points of the string, ρ is the linear mass, T is the tension, and A_0', A_1', A_2' and B' are constants.

It is easily seen that the problem of characteristic vibrations of such a string reduces to the boundary value problem (1.1), if we assume that

$$x=\frac{\xi}{l}, \quad \lambda^2=\frac{\rho\omega^2 l^2}{T}, \quad A_0=\frac{A_0'}{l}, \quad A_1=\frac{A_1'T}{\rho l^3}, \quad A_2=\frac{A_2'T^2}{\rho^2 l^5}, \quad B=\frac{B'}{l} \qquad (2.3)$$

Here ω is the characteristic frequency of vibrations. Given initial conditions

$$t=0, \quad y=0, \quad \frac{\partial y}{\partial t}=f(\xi) \tag{2.4}$$

we obtain from (2.1) and (2.4)

$$t=0, \quad \frac{\partial^2 y}{\partial t^2}=0, \quad \frac{\partial^3 y}{\partial t^3}=\frac{T}{\rho}\frac{d^2 f}{d\xi^2} \tag{2.5}$$

3. Let us apply Laplace transformation to (2.1) and (2.2), taking the initial conditions (2.4) and (2.5) into account. This will give us

$$\frac{d^2 Y}{d\xi^2}-\frac{\rho s^2}{T}Y=-\frac{\rho}{T}f(\xi) \tag{3.1}$$

$$\xi=0, \quad (A_0'-A_1's^2+A_2's^4)\frac{dY}{d\xi}+(A_1'-s^2A_2')\frac{df}{d\xi}-\frac{T}{\rho}A_2'\frac{d^3 f}{d\xi^3}=Y$$

$$\xi=l, \quad B'\frac{dY}{d\xi}=Y \tag{3.2}$$

where $Y(\xi, s)$ is the transform of $y(\xi, t)$. Putting $\omega^2=-s^2$ and changing to the dimensionless coordinate x, we obtain as a result

$$\frac{d^2 Y}{dx^2}+\lambda^2 Y=-\frac{\rho l^2}{T}f(x), \tag{3.3}$$

$$x=0, \ c_1(\lambda)\frac{dY}{dx}+\frac{\rho l^2}{T}\left[c_2(\lambda)\frac{df}{dx}-A_2\frac{d^3 f}{dx^3}\right]=Y \qquad \begin{pmatrix} c_1(\lambda)=A_0+A_1\lambda^2+A_2\lambda^4 \\ c_2(\lambda)=A_1+A_2\lambda^2 \end{pmatrix}$$

$$x=1, \quad B\frac{dY}{dx}=Y \tag{3.4}$$

Let us represent the general solution in the form $Y=Y_1+Y_2$ where Y_1 is the solution of a nonhomogeneous equation

$$\frac{d^2 Y_1}{dx^2}+\lambda^2 Y_1=-\frac{\rho l^2}{T}f(x) \tag{3.5}$$

with homogeneous boundary conditions

$$x=0, \quad c_1(\lambda)\frac{dY_1}{dx}=Y_1; \qquad x=1, \quad B\frac{dY_1}{dx}=Y_1 \tag{3.6}$$

while Y_2 is the solution of a homogeneous equation

$$\frac{d^2 Y_2}{dx^2}+\lambda^2 Y_2=0$$

with nonhomogeneous boundary conditions

$$x=0, \quad c_1(\lambda)\frac{dY_2}{dx}-Y_2=-\frac{\rho l^2}{T}\left[c_2(\lambda)\frac{df}{dx}-A_2\frac{d^3 f}{dx^3}\right]; \quad x=1, \quad B\frac{dY_2}{dx}=Y_2 \tag{3.7}$$

We then obtain

$$Y_1=\frac{\rho l^2}{T}\frac{1}{2\lambda}\frac{1}{\lambda c_1(\lambda)(\cos\lambda+B\lambda\sin\lambda)-B\lambda\cos\lambda+\sin\lambda}\times \tag{3.8}$$

$$\times\left\{\lambda c_1(\lambda)\left[\int_x^1 f(\zeta)\sin\lambda(1+x-\zeta)\,d\zeta+\int_0^1 f(\zeta)\sin\lambda(1-x-\zeta)\,d\zeta+\right.\right.$$

$$+\int_0^x f(\zeta)\sin\lambda(1-x+\zeta)\,d\zeta-B\lambda\left(\int_x^1 f(\zeta)\cos\lambda(1+x-\zeta)\,d\zeta+\right.$$

$$\left.\left.+\int_0^1 f(\zeta)\cos\lambda(1-x-\zeta)\,d\zeta+\int_0^x f(\zeta)\cos\lambda(1-x+\zeta)\,d\zeta\right)\right]-$$

$$-\int_x^1 f(\zeta)\cos\lambda(1+x-\zeta)\,d\zeta+\int_0^1 f(\zeta)\cos\lambda(1-x-\zeta)\,d\zeta-\int_0^x f(\zeta)\cos(1-x+\zeta)\,d\zeta+$$

$$+B\lambda\left[-\int_x^1 f(\zeta)\sin\lambda(1+x-\zeta)\,d\zeta+\int_0^1 f(\zeta)\sin\lambda(1-x-\zeta)\,d\zeta-\right.$$

$$\left.\left.-\int_0^x f(\zeta)\sin\lambda(1-x+\zeta)\,d\zeta\right]\right\}$$

$$Y_2=-\frac{\rho l^2}{T\lambda}\,\frac{\lambda\,[B\lambda\cos\lambda(1-x)-\sin\lambda(1-x)]\,[c_2(\lambda)\,(df/dx)_{x=0}-A_2\,(d^3f/dx^3)_{x=0}]}{\lambda c_1(\lambda)(\cos\lambda+B\lambda\sin\lambda)-B\lambda\cos\lambda+\sin\lambda} \qquad (3.9)$$

4. Function $Y = Y_1 + Y_2$ is a meromorphic function of a complex variable λ, simple poles of which are given by

$$\lambda c_1(\lambda)(\cos\lambda+B\lambda\sin\lambda)-B\lambda\cos\lambda+\sin\lambda=0 \qquad (4.1)$$

The above equation also yields the eigenvalues of the boundary value problem (1.1). Equation (4.1) has an enumerable infifity of real and finite number of imaginary and complex roots. Real and imaginary axes of the complex plane λ are the axes of symmetry of these roots.

It can easily be shown that the expansion of Y into simple fractions is

$$Y(x,\lambda)=\sum_{(m)}\frac{\mathrm{res}_{\lambda_m}Y(x,\lambda)}{\lambda-\lambda_m} \qquad (4.2)$$

Summation in (4.2) is performed over all poles of the complex plane λ

$$\mathrm{res}_{\lambda_m}Y(x,\lambda)=-\frac{\rho l^2}{T}\,\frac{D_1+\lambda_m\,[c_2(\lambda_m)(df/dx)_{x=0}-A_2\,(d^3f/dx^3)_{x=0}]}{2\lambda_m\,\{D_2+\lambda_m^2\,[c_2(\lambda_m)+\lambda_m^2A_2]\}}\,y_m(x) \qquad (4.3)$$

$$\left(D_1=\int_0^1 f(\zeta)\,y_m(\zeta)\,d\zeta,\qquad D_2=\int_0^1 y_m^2(\zeta)\,d\zeta\right)$$

Here $y_m(x)$ is the eigenfunction of (1.1) corresponding to the eigen number λ_m.

By virtue of the symmetry of the eigenvalues λ_m with respect to imaginary axis of the complex plane λ and since $\mathrm{res}_{-\lambda m}Y(x,\lambda)=-\mathrm{res}_{\lambda m}Y(x,\lambda)$, expansion (4.2) assumes the form

$$Y(x,\lambda)=\sum_{(m)}\frac{2\lambda_m\,\mathrm{res}_{\lambda_m}Y(x,\lambda)}{\lambda^2-\lambda_m^2} \qquad (4.4)$$

where the summation is performed over the poles of the right-hand semi-plane λ, including the positive part of the imaginary semi-axis.

5. Applying to (4.4) the inverse Laplace transformation and taking into account the fact that $\lambda^2=-(\rho l^2/T)\,s^2$, we obtain

$$y(x,t)=-\left(\frac{T}{\rho l^2}\right)^{1/2}\sum_{(m)}2\lambda_m\,\mathrm{res}_{\lambda_m}Y(x,\lambda)\sin\left[\left(\frac{T}{\rho l^2}\right)^{1/2}\lambda_m t\right] \qquad (5.1)$$

In accordance with the initial conditions (2.4), we have

$$f(x)=\sum_{(m)}\frac{D_1+\lambda_m\,[c_2(\lambda_m)\,(df/dx)_{x=0}-A_2\,(d^3f/dx^3)_{x=0}]}{D_2+\lambda_m^2\,[c_2(\lambda_m)+\lambda_m^2A_2]}\,y_m(x) \qquad (5.2)$$

Thus we have shown that the system of eigenfunctions of the boundary problem (1.1) is complete and that the expansion of an arbitrary function $f(x)$ into the series (5.2), is unique.

6. To illustrate the application of obtained results, we shall consider a plane problem of hydromechanics. It will be a problem on small, axially-symmetric, characteristic oscillations of an elastic inertialess frame with an ideal incompressible fluid in the absence of gravity (see Fig.1). Let Φ be the velocity potential of the fluid, p its pressure, ρ its density, w_1 and w_2 the deflections and let E_1 and E_2 be the flexural rigidities of the bars 1 and 2, respectively, Positive directions of w_1 and w_2 are indicated in the figure. We have the following equations of motion of the fluid

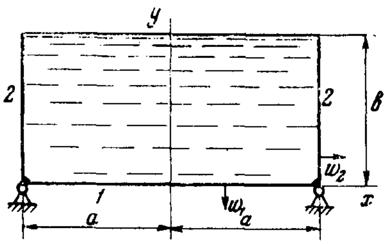

Fig. 1

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0, \qquad p=\rho\frac{\partial\Phi}{\partial t} \tag{6.1}$$

and the boundary conditions

$$x=0,\ \frac{\partial\Phi}{\partial x}=0; \qquad x=a,\ -\frac{\partial\Phi}{\partial x}=w_2^{\cdot}$$

$$y=0,\ \frac{\partial\Phi}{\partial y}=w_1^{\cdot}; \qquad y=b,\ \Phi=0 \tag{6.2}$$

Here and in the following, a dot denotes differentiation with respect to time.

Equations of motion of the bars are

$$EI_1\frac{\partial^4 w_1^{\cdot}}{\partial x^4}=[p^{\cdot}]_{y=0}; \qquad EI_2\frac{\partial^4 w_2^{\cdot}}{\partial y^4}=[p^{\cdot}]_{x=a} \tag{6.3}$$

and their boundary conditions are

$$x=0,\quad \frac{\partial w_1^{\cdot}}{\partial x}=\frac{\partial^3 w_1^{\cdot}}{\partial x^3}=0; \qquad x=a,\quad w_1^{\cdot}=0$$

$$\left[\frac{\partial w_1^{\cdot}}{\partial x}\right]_{x=a}=\left[\frac{\partial w_2^{\cdot}}{\partial y}\right]_{y=0} \qquad EI_1\left[\frac{\partial^2 w_1^{\cdot}}{\partial x^2}\right]_{x=a}=EI_2\left[\frac{\partial^2 w_2^{\cdot}}{\partial y^2}\right]_{y=0}$$

$$y=0,\quad w_2^{\cdot}=0; \qquad y=b,\quad \frac{\partial^2 w_2^{\cdot}}{\partial y^2}=\frac{\partial^3 w_2^{\cdot}}{\partial y^3}=0 \tag{6.4}$$

The velocity potential of the fluid satisfying part of the boundary conditions (6.2), can be written in the form

$$\Phi=\left[\sum_{(m)}A_m\sinh\lambda_m(b-y)\cos\lambda_m x+\sum_{(k)}B_k\cosh\mu_k x\sin\mu_k(b-y)\right]\sin\omega t \tag{6.5}$$

Here λ_m and $\cos\lambda_m x$ are the eigenvalues and eigenfunctions of the following boundary value problem

$$\frac{d^2X}{dx^2}+\lambda^2X=0, \quad x=0,\quad \frac{dX}{dx}=0; \qquad x=a,\quad EI_2\lambda^4\frac{dX}{dx}=\rho\omega^2X$$

while μ_k and $\sin\mu_k(b-y)$ are the eigenvalues and eigenfunctions of

$$\frac{d^2Y}{dy^2}+\mu^2Y=0, \qquad y=0,\quad EI_1\mu^4\frac{dY}{dy}=-\rho\omega^2Y; \qquad y=b,\quad Y=0$$

We shall utilize the method first proposed by Leibenzon in [2], to obtain the conditions of simultaneity of velocities of motion of the fluid and the elastic vessel. We shall consider pressure of the fluid as an external load on the vessel. Solving the equations of forced vibrations of the bars (6.3), we find

$$w_1 = -\frac{\rho\omega^2}{EI_1}\Big[C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4 + \\ + \sum_{(m)} A_m \frac{\sinh\lambda_m b}{\lambda_m^4}\cos\lambda_m x + \sum_{(k)} B_k \frac{\sin\mu_k b}{\mu_k^4}\cosh\mu_k x\Big] \quad (6.6)$$

$$w_2 = -\frac{\rho\omega^2}{EI_2}\Big[D_1\frac{(b-y)^3}{6} + D_2\frac{(b-y)^2}{2} + D_3(b-y) + D_4 + \\ + \sum_{(m)} A_m \frac{\cos\lambda_m a}{\lambda_m^4}\sinh\lambda_m(b-y) + \sum_{(k)} B_k \frac{\cosh\mu_k a}{\mu_k^4}\sin\mu_k(b-y)\Big] \quad (6.7)$$

Here C_j and D_j $(j = 1, 2, 3, 4)$ are constants of integration. Functions of time are, in (6.6) and (6.7) and subsequent equations, neglected.

Inserting the velocities of deflection of bars (6.6) and (6.7) into the boundary conditions (6.2) and taking into account the equations defining the eigen numbers λ_m and μ_k, we obtain the following functional expressions:

$$C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4 = \sum_{(m)} A_m\left[\frac{EI_1}{\rho\omega^2}\lambda_m\cosh\lambda_m b - \frac{\sinh\lambda_m b}{\lambda_m^4}\right]\cos\lambda_m x \quad (6.8)$$

$$D_1\frac{(b-y)^3}{6} + D_2\frac{(b-y)^2}{2} + D_3(b-y) + D_4 = \quad (6.9)$$

$$= \sum_{(k)} B_k\left[\frac{EI_2}{\rho\omega^2}\mu_k\sinh\mu_k a - \frac{\cosh\mu_k a}{\mu_k^4}\right]\sin\mu_k(b-y)$$

which, together with former results, can yield all A_m in terms of C_j and all B_k in terms of D_j $(j = 1, 2, 3, 4)$. Substituting these into (6.6) and (6.7) with the boundary conditions (6.4) being satisfied, we can obtain the system of linear equations homogeneous in C_j and D_j. By equating the determinant of this system to zero, we obtain the frequency equation which will contain infinite sums of rapidly converging series. Their strong convergence can be explained by the fact, that no differentiation which would weaken the convergence, is performed in the course of solution of the problem.

Asymptotic values of characteristic frequencies can be found from the solution of the following system of transcendental equations:

$$\frac{EI_2}{\rho\omega^2 a^5}(\lambda a)^5\tan\lambda a + 1 = \frac{EI_1}{\rho\omega^2 b^5}(\lambda b)^5\coth\lambda b - 1 = 0 \quad (6.10)$$

The above system defines the distribution of asymptotes of a moromorphic frequency function corresponding to the frequency equation of our problem.

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